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Rigorous derivation of the NLS in magnetic films

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Abstract

The nonlinear Schrödinger (NLS) equation that gives an account of 'temporal' envelope soliton propagation in magnetic thin films is derived, using the rigorous asymptotic method of multiscale expansions. Magnetostatic backward volume waves are considered and inhomogeneous exchange is neglected. New mathematical features concerning multiscale expansions are found: both a propagating second harmonic term and the usual non-propagating one arise. The secular-type terms that arise in the transverse direction are no longer forbidden. The dispersion coefficient of the obtained NLS equation is not equal to the so-called group velocity dispersion, despite being a general law in bulk media. The nonlinear coefficient is in quite good agreement with the result of previous computations for physically relevant values of the parameters.

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1. Introduction

The interest in envelope solitons in magnetic media is twofold: on one hand, the mathematical properties of the so-called Maxwell–Landau model are of major importance for the nonlinear wave propagation theory. On the other hand, soliton propagation in thin films is investigated experimentally with a much more accurate description than, for example, optical solitons. Further, it is very promising for applications.

The theory of one-dimensional envelope solitons in any case rests on the so-called (cubic) nonlinear Schrödinger (NLS) equation. Studies have been developed to give a rigorous mathematical derivation of this equation and of some of its generalizations, as asymptotic reductions of the Maxwell–Landau model [1–3]. Proof of the convergence of this asymptotic has been given, from the viewpoint of pure mathematics, in the frame of the multiscale expansion formalism [4]. The latter thus appears to be the most rigorous way to derive the NLS equation. These theoretical works essentially concern the propagation of electromagnetic waves in bulk media only, taking retardation into account. Such waves are sometimes called magnetic polaritons.

Besides this, many experiments have been done showing the formation of solitons and dark solitons [5–7], their collapse or spreading out in two dimensions [8] and so on. Almost all experiments are performed on thin films of yttrium iron garnet (YIG) in a situation where retardation is negligible. The considered waves are called magnetostatic spin waves (MSWs). The film thickness is of the order of the magnitude of the wavelength, or even less. In such a very thin waveguide, the linear dispersion relation strongly differs from that of the bulk material [9]. It is expected that the nonlinear properties are also strongly modified. That is why a multiscale theory taking into account the waveguide properties, and the smallness of its thickness in an explicit way, appears to be necessary in order to give a rigorous derivation of the NLS equation commonly used for the description of the experimentally observed solitons [10–11] and more accurate expressions of its coefficients.

This is the aim of the present paper. Several new features are found and some important deviations from the usual NLS-type multiscale expansions, valid in a bulk medium, are observed. The second harmonic is no longer a simple non-propagating term, with the same velocity as the fundamental, but contains an additional term propagating at the proper second-harmonic velocity. From a more technical point of view, an important characteristic of the multiscale theory in bulk media is that it involves the vanishing of the so-called secular terms, which grow linearly with regard to the variable. In thin films, and in the transverse direction, linear growth is allowed because of finite film thickness.

A very remarkable result is that the dispersion coefficient, the coefficient of the second derivative in the NLS equation, differs from the commonly admitted value of the group velocity dispersion. The expression valid in the bulk medium, using a second derivative of the dispersion relation, is still valid when the film is relatively large. But it can be strongly modified for the fundamental mode in a very thin film.

This paper is organized as follows: after the introduction section, the starting model and hypothesis are set out in section 2. Then the derivation is presented order by order: the linear dispersion relation that yields the first order of the perturbative scheme is derived in section 3. The second order involves some interesting features about nonlinear terms and is presented in section 4. The condition that ensures wave packet propagation at the group velocity is derived in section 5 and the nonlinear evolution equation (NLS) in section 6. The coefficients of this equation are studied in section 7 and section 8 yields a conclusion. A large part of the technical detail of the derivation has been put in several appendices.

2. Setting the problem

2.1. The Maxwell–Landau model for a thin film

We assume that the magnetic medium fills the region of the space between $z = 0$ and $z = L$ (figure 1). The upper region is denoted by the letter a, the region below the film by b and the inside of the film by i. The Maxwell equations in the magnetostatic limit read as

$$\vec{\nabla} \wedge \vec{H} = \vec{0} \tag{1}$$

$$\vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0. \tag{2}$$

Note that the second-order form of the Maxwell equations (or wave equation) introduces incorrect solutions in this magnetostatic limit. In contrast, the divergence equation, which is not needed when retardation is taken into account, is necessary here. Inside the film, the magnetization $\vec{M}$ obeys the so-called Landau–Lifschitz equation which reads

$$-\partial_t \vec{M} = \vec{M} \wedge \vec{H}. \tag{3}$$

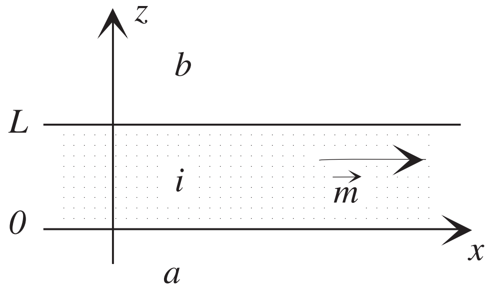

Figure 1. Geometry of the problem and coordinate frame.

The anisotropy and inhomogeneous exchange are neglected. The demagnetizing field does not appear when $\vec{H}$ represents the magnetic field that really exists in the medium.

$\vec{M}$ is zero outside the magnetic film. The boundary conditions are magnetostatic: the tangential component of $\vec{H}$ and the normal component of $\vec{B} = \vec{H} + \vec{M}$ are continuous at the surface of the film $z = 0$ and $z = L$. Further, $\vec{H}$ tends to some constant applied field $\vec{H}_0$ as z tends to infinity.

2.2. A scaling

Let us now introduce multiple scales. ε is some small perturbative parameter that will, as usual, give an account of the smallness of both the signal amplitude and its spectral width. Precisely, the fields are expanded as

$$\vec{H} = \sum_{n\geqslant 0, |p|\leqslant n} \varepsilon^n \mathrm{e}^{\mathrm{i}p\phi} \vec{H}_n^p \tag{4}$$

$$\vec{M} = \sum_{n\geqslant 0, |p|\leqslant n} \varepsilon^n \mathrm{e}^{\mathrm{i}p\phi} \vec{M}_n^p. \tag{5}$$

The fundamental phase ϕ reads as

$$\phi = kx - \omega t \tag{6}$$

corresponding to a propagation along the x-axis. The $\vec{H}_n^p$ and $\vec{M}_n^p$ are functions of the slow variables $\xi = \varepsilon(x - Vt)$, $\tau = \varepsilon^2 t$ and z. They vanish as $\xi \longrightarrow -\infty$, except the uniform constant applied field $\vec{H}_0^0$ and the corresponding magnetization $\vec{M}_0^0$. This is the standard ansatz used for the derivation of the NLS equation in one dimension. Note that the z variable that gives an account of the variations inside the film has the order ε^0, which means that the order of magnitude of the film thickness is assumed to be the same as the wavelength. In many experiments, the thickness is even smaller, but the expansion would become singular with such an assumption. This would give rise to further mathematical difficulties; and therefore the corresponding situation is left for further investigation. We assume for the sake of simplicity that the wave is not modulated transversely. Then it is easily seen from equation (1) that $\vec{H}^y$ is uniform and equation (1) reduces to

$$\partial_z H_x = \partial_x H_z. \tag{7}$$

2.3. The steady state

The zero order of the perturbative expansion is some steady state, determined by some uniform and constant applied field $\vec{H}_0$, setting the film magnetization to its saturation value m. Outside the magnetic medium (regions a and b) there is only the field $\vec{H}_0^0$. The Maxwell equations show that it is a constant, equal to the exterior field $\vec{H}_0$.

In the magnetic medium (region i), $\vec{H}_0^0$ and $\vec{M}_0^0$ are constants with regard to z, and are collinear. The boundary conditions yield three equations for four unknowns: the failing condition is the statement that the norm of $\vec{M}_0^0$ is equal to the saturation magnetization m. For the sake of simplicity, we choose the particular case of wave propagation along the direction of the magnetization. Then

$$\vec{M}_0^0 = \vec{m} = \begin{pmatrix} m \\ 0 \\ 0 \end{pmatrix} \qquad \vec{H}_0^0 = \alpha\vec{m} \tag{8}$$

where the parameter $\alpha = H_0/m$ gives an account of the external field strength. Avoid confusion with the inhomogeneous exchange interaction coefficient, which is frequently denoted by the same letter α, e.g. in [11], but neglected here. MSWs propagating parallel to the external field have a negative group velocity. This well known result [9] is retrieved by our formalism (see below). The waves are called magnetostatic backward volume waves (MSBVWs) in this case.

Before we solve the perturbative scheme, some preliminary reduction of the equations is needed. The fact that H^y is uniform, together with the boundary conditions and the choice of the external field, yields $\vec{H}_n^{p,y} = 0$ for any p and n. Then the Maxwell equations reduce to (2) and (7) with $\vec{M} = \vec{0}$ outside the magnetic medium (regions a and b). These equations, together with (3), are expanded into a power series of ε. The perturbative scheme is then solved by setting the coefficients of each power of ε on both sides of the equation equal to each other. For convenience, we define nonlinear terms $\vec{\Xi}_n^p$ as follows:

$$\varepsilon^2\vec{\Xi}_2^p + \varepsilon^3\vec{\Xi}_3^p + \cdots = \sum_{r+s=p}\left(\varepsilon\vec{M}_1^r + \cdots\right) \wedge \left(\varepsilon\vec{H}_1^s + \cdots\right). \tag{9}$$

3. Retrieving the linear waveguide dispersion law

3.1. The evanescent waves outside the magnetic medium

We solve equations (2)–(7) at order ε^1 with zero magnetization in the regions a and b outside the magnetic film. It is easily seen from these equations that the components of $\vec{H}_1^1$ are linear combinations of e^{kz} and e^{-kz}. Assuming k positive and taking into account the vanishing of $\vec{H}$ at infinity, it is found that:

$$\vec{H}_1^1 = A_1^1\begin{pmatrix} 1 \\ 0 \\ \mathrm{i} \end{pmatrix} e^{-kz} \qquad \text{for } z > L \qquad \text{and} \qquad B_1^1\begin{pmatrix} 1 \\ 0 \\ -\mathrm{i} \end{pmatrix} e^{+kz} \qquad \text{for } z < 0 \tag{10}$$

where A_1^1 and B_1^1 are complex-valued functions of the slow variables.

3.2. Inside the magnetic medium

In the region i, $\vec{M}_1^1$ does not vanish and the Landau–Lifschitz equation (3) must be taken into account. It is first seen that $M_1^{1,x} = 0$. The four other equations can be written in the matrix form

$$\mathcal{L}_1 \begin{pmatrix} H_1^{1,x} \\ H_1^{1,z} \\ M_1^{1,y} \\ M_1^{1,z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

where the linear operator $\mathcal{L}_p$ is written as

$$\mathcal{L}_p = \begin{pmatrix} \mathrm{i}pk & \partial_z & 0 & \partial_z \\ \partial_z & -\mathrm{i}pk & 0 & 0 \\ 0 & m & \mathrm{i}p\omega & -\alpha m \\ 0 & 0 & \alpha m & \mathrm{i}p\omega \end{pmatrix}. \tag{12}$$

Because this operator appears at each order, it is useful to solve the general system

$$\mathcal{L}_p \begin{pmatrix} H^x \\ H^z \\ M^y \\ M^z \end{pmatrix} = \begin{pmatrix} T^x \\ T^z \\ U^y \\ U^z \end{pmatrix} \tag{13}$$

for any rhs member (T^x, T^z, U^y, U^z). This is done in appendix A. The resolution of system (13) rests on the scalar differential equation

$$\left(\partial_z^2 + q_p^2\right) H^z = \Sigma \tag{14}$$

where Σ is some linear functional of the rhs member of the system, given by formula (83) in appendix A, and

$$q_p = pk \left(\frac{p^2\omega^2 - \alpha^2 m^2}{\alpha(\alpha+1)m^2 - p^2\omega^2} \right)^{1/2}. \tag{15}$$

Note that q_p is not necessarily real. Let us call H_s^z some particular solution of equation (14). The general solution of the latter is written as

$$H^z = R\mathrm{e}^{\mathrm{i}q_p z} + S\mathrm{e}^{-\mathrm{i}q_p z} + H_s^z. \tag{16}$$

In the same way, the other components of $\vec{H}$ and $\vec{M}$ are expressed as the sum of two terms. One of these terms is some particular solution of the complete system, which is a linear combination of H_s^z and the rhs member components. It is denoted below by the subscript s, and is referred to as the inhomogeneous part of the solution. The other term is the general solution of the homogeneous system, which is a linear combination of $R\mathrm{e}^{\mathrm{i}q_p z}$ and $S\mathrm{e}^{-\mathrm{i}q_p z}$. It is referred to as the homogeneous part of the solution below. Explicit formulae are given in appendix A.

System (13) is homogeneous, thus $H_{1,s}^{1,z} = 0$. The solution is found from the above-mentioned general formulae, and for $0 < z < L$ reads as

$$\vec{H}_1^1 = \begin{pmatrix} \frac{k}{q}\left(R_1^1\mathrm{e}^{\mathrm{i}qz} - S_1^1\mathrm{e}^{-\mathrm{i}qz}\right) \\ 0 \\ R_1^1\mathrm{e}^{\mathrm{i}qz} + S_1^1\mathrm{e}^{-\mathrm{i}qz} \end{pmatrix} \tag{17}$$

$$\vec{M}_1^1 = \frac{m}{\alpha^2 m^2 - \omega^2}\left(R_1^1\mathrm{e}^{\mathrm{i}qz} + S_1^1\mathrm{e}^{-\mathrm{i}qz}\right) \begin{pmatrix} 0 \\ -\mathrm{i}\omega \\ \alpha m \end{pmatrix} \tag{18}$$

with $q = q_1$ is defined by equation (15) so that

$$\omega^2 = \alpha\left(\alpha + \frac{q^2}{k^2+q^2}\right) m^2. \tag{19}$$

Equation (19) is well known [9]; it is the dispersion relation of MSW in an infinite medium, satisfied by the wave vector $(k, 0, q)$. The waveguide dispersion relation is determined by the conditions at the surfaces $z = 0$ and $z = L$ of the film.

3.3. Boundary conditions

The magnetostatic boundary conditions, i.e. the continuity of the parallel components $H_1^{1,x}$, $H_1^{1,y}$ and of the transverse component $\left(H_1^{1,z} + M_1^{1,z}\right)$, must be satisfied at the boundaries $z = 0$ and $z = L$ of the magnetic film. Using the expressions (10), (17) and (18) of the fields, they are reduced to the following linear system for $\left(A_1^1, B_1^1, R_1^1, S_1^1\right)$:

$$\begin{aligned} B_1^1 &= \frac{k}{q}\left(R_1^1 - S_1^1\right) \\ -\mathrm{i}B_1^1 &= \frac{-k^2}{q^2}\left(R_1^1 + S_1^1\right) \\ A_1^1\mathrm{e}^{-kL} &= \frac{k}{q}\left(R_1^1\mathrm{e}^{\mathrm{i}qL} - S_1^1\mathrm{e}^{-\mathrm{i}qL}\right) \\ \mathrm{i}A_1^1\mathrm{e}^{-kL} &= \frac{-k^2}{q^2}\left(R_1^1\mathrm{e}^{\mathrm{i}qL} + S_1^1\mathrm{e}^{-\mathrm{i}qL}\right). \end{aligned} \tag{20}$$

As above, the resolution of a general version of system (20) is useful. It is done in appendix B. Note that, at the present order, the system (20) is homogeneous. A wave can thus propagate only if the rank of the system is at most 3. This condition yields the dispersion relation of the guide

$$2q\ \cot qL = \frac{1}{k}(q^2 - k^2). \tag{21}$$

Here we retrieve this well known result [9]. The complete solution at order 1 is then obtained using the general formulae of appendix B for a homogeneous system. Inside the magnetic medium it reads as

$$\vec{H}_1^1 = \begin{pmatrix} -\mathrm{i}k\varphi_2 \\ 0 \\ q\varphi_1 \end{pmatrix} g \tag{22}$$

$$\vec{M}_1^1 = \begin{pmatrix} 0 \\ -\mathrm{i}\omega \\ \alpha m \end{pmatrix} \frac{mqg\varphi_1}{\alpha^2 m^2 - \omega^2}. \tag{23}$$

Here g is some function of the slow variables ξ and τ to be determined. The mode profiles are determined by the functions φ_1 and φ_2 defined by

$$\varphi_1 = q\ \cos qz + k\ \sin qz \tag{24}$$

$$\varphi_2 = k\ \cos qz - q\ \sin qz. \tag{25}$$

Some properties of these profiles are given in appendix C.1.

4. The nonlinear terms

4.1. At second order

In the magnetic medium and at the second order in the perturbation theory, equations (2), (3) and (7), for the pth harmonic, yield

$$M_2^{p,x} = \frac{1}{\mathrm{i}p\omega}\left(\Xi_2^{p,x} - V\partial_\xi M_1^{p,x}\right) \tag{26}$$

and

$$\mathcal{L}_p \begin{pmatrix} H_2^{p,x} \\ H_2^{p,z} \\ M_2^{p,y} \\ M_2^{p,z} \end{pmatrix} = \begin{pmatrix} T_2^{p,x} \\ T_2^{p,z} \\ U_2^{p,y} \\ U_2^{p,z} \end{pmatrix}. \tag{27}$$

The rhs member of equation (27) has the following expression:

$$\begin{aligned} T_2^{p,x} &= \frac{-k}{\omega}\left(\Xi_2^{p,x} - V\partial_\xi M_1^{p,x}\right) - \partial_\xi\left(H_1^{p,x} + M_1^{p,x}\right) \\ T_2^{p,z} &= \partial_\xi H_1^{p,z} \\ U_2^{p,j} &= \Xi_2^{p,j} - V\partial_\xi M_1^{p,j} \qquad \text{for } j = y, z. \end{aligned} \tag{28}$$

Except for $p = \pm 1$, system (27) has a rank 4. It is not homogeneous and admits a nonzero solution only if the nonlinear term $\vec{\Xi}_2^p$ is not zero.

The nonzero nonlinear terms at this order are

$$\vec{\Xi}_2^2 = \vec{M}_1^1 \wedge \vec{H}_1^1 \tag{29}$$

$$\vec{\Xi}_2^0 = \vec{M}_1^1 \wedge \vec{H}_1^{1,*} + \vec{M}_1^{1,*} \wedge \vec{H}_1^1 \tag{30}$$

and $\vec{\Xi}_2^{-2} = \vec{\Xi}_2^{2,*}$ (an asterisk denotes complex conjugation). The term $\vec{\Xi}_2^2$ gives rise to some non-vanishing second harmonic. We study it in this section. $\vec{\Xi}_2^0$ gives rise to a mean value term or the rectified field studied in section 5. They are computed using the results of order 1, namely formulae (22) and (23), and read

$$\vec{\Xi}_2^2 = \frac{mq}{\alpha^2 m^2 - \omega^2} g^2 \begin{pmatrix} -\mathrm{i}\omega q f_1 \\ -\mathrm{i}\alpha m k f_2 \\ \omega k f_2 \end{pmatrix} \tag{31}$$

$$\vec{\Xi}_2^0 = \frac{-2m\omega k q}{\alpha^2 m^2 - \omega^2} |g|^2 \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix} \tag{32}$$

where $f_1 = \varphi_1^2$ and $f_2 = \varphi_1\varphi_2$. Some useful properties of f_1 and f_2 are given in appendix C.2.

4.2. *The second harmonic directly emitted by the fundamental*

The x-component of the magnetization density is computed directly from expressions (26) and (31). It reads

$$M_2^{2,x} = \frac{-mq^2}{2(\alpha^2 m^2 - \omega^2)} g^2 f_1. \tag{33}$$

The system (27) is then solved using the formulae proved in appendix A for the general case (13), as detailed in appendix D.1.1. A particular solution of system (27), that is the inhomogeneous part of the second harmonic, denoted by the subscript s is first computed. The key point of this computation is that of the component $H_{2,s}^{2,z}$, a particular solution of equation (14), with an rhs member $\Sigma_2^2 = \nu f_2$, where ν is a constant given by equation (112) in appendix D.1.1. This solution is sought under the form

$$H_{2,s}^{2,z} = w f_2. \tag{34}$$

It is found that

$$w = \frac{\nu}{q_2^2 - 4q^2}. \tag{35}$$

This implies that $q_2 \neq 2q$, which means that the phase-matching for second-harmonic generation is not realized. If this were realized, the interaction between fundamental and second harmonic would be resonant, and an efficient conversion to second harmonic would produce a non-vanishing amplitude for $\vec{H}_1^2$, the second harmonic at order 1. The assumption made in our ansatz, that this term is zero, is not consistent if phase matching is realized. Because it is not the case in real physical situations, we assume that $q_2 \neq 2q$ is satisfied. Finally, we get

$$H_{2,s}^{2,z} = \frac{-\mathrm{i}\alpha mkq}{2\omega^2} g^2 f_2. \tag{36}$$

The other components of this field are written in appendix A (equations (84), (86) and (88)). Their inhomogeneous parts read

$$H_{2,s}^{2,x} = \frac{mq^2}{2(\alpha^2 m^2 - \omega^2)} g^2 \frac{k^2 + q^2}{2} + \frac{\alpha mk^2}{2\omega^2} g^2 \mu \cos q(2z - L) \tag{37}$$

$$M_{2,s}^{2,y} = \frac{-\alpha m^2 kq}{\omega(\alpha^2 m^2 - \omega^2)} g^2 f_2 \tag{38}$$

$$M_{2,s}^{2,z} = \frac{-\mathrm{i}mkq(\alpha^2 m^2 + \omega^2)}{2\omega^2(\alpha^2 m^2 - \omega^2)} g^2 f_2. \tag{39}$$

As is usual in multiscale expansion, these quantities describe some non-propagating second-harmonic term that moves at the exact speed of the fundamental.

4.3. A true second harmonic

Outside the magnetic medium, the expression for the second harmonic $p = 2$ is the same as at order 1, equation (10), with k replaced by $2k$ and amplitudes A_2^2, B_2^2.

The equations giving account for the boundary conditions, at second order and for the second harmonic, yield some linear system, analogous to (20) but inhomogeneous. It is solved using the formulae of appendix B, as detailed in appendix D.1.2. It allows computation of the homogeneous parts of $H_2^{2,x}$ and $H_2^{2,z}$, which read respectively as

$$\frac{2k}{q_2}\left(R_2^2 \mathrm{e}^{\mathrm{i}q_2 z} - S_2^2 \mathrm{e}^{-\mathrm{i}q_2 z}\right) = \frac{4kY}{q_2} g^2 \cos q_2 \left(z - \frac{L}{2}\right) \tag{40}$$

$$R_2^2 \mathrm{e}^{\mathrm{i}q_2 z} + S_2^2 \mathrm{e}^{-\mathrm{i}q_2 z} = 2\mathrm{i}Y g^2 \sin q_2 \left(z - \frac{L}{2}\right). \tag{41}$$

The constant Y is given by the following expression:

$$Y = \frac{-mq^2 q_2^2[q^2\omega^2 - k^2(\alpha^2 m^2 + \omega^2)]}{16\omega^2 k(\alpha^2 m^2 - \omega^2)\left[2k \sin q_2 \frac{L}{2} + q_2 \cos q_2 \frac{L}{2}\right]}. \tag{42}$$

Adding the homogeneous part of $H_2^{2,z}$, computed above, to its inhomogeneous part $H_{2,s}^{2,z}$ given by (36) yields

$$H_2^{2,z} = 2\mathrm{i}Y g^2 \sin q_2 \left(z - \frac{L}{2}\right) - \frac{\mathrm{i}\alpha mkq}{2\omega^2} f_2. \tag{43}$$

The first term in (43) involves q_2, which is the z-component of the wave vector satisfying the dispersion relation in a bulk medium (15) with the pulsation 2ω and an x-component of the wave vector $2k$. Note that q_2 can be purely imaginary. In this case, Y is also imaginary and the sine and cosine functions in the above formulae behave like hyperbolic sine and cosine. The second term oscillates with the z-component of the wave vector equal to $2q$. This feature is particular to the waveguide situation. Indeed, in an infinite medium, if no phase matching occurs, the second harmonic does not propagate with its proper phase and group velocity, but only accompanies the fundamental having the same velocities. In other words, its phase dependence involves $2\vec{k}(\omega)$ and not $\vec{k}(2\omega)$. Here, in contrast, q_2 appears. The reflection of the non-propagating second harmonic generates a new component that, unlike what happens in infinite media, propagates inside the nonlinear medium with its own velocity, at least along the transverse direction. However, q_2 is not a waveguide mode; the equation analogous to (21) is not satisfied by q_2: $D_2 \neq 0$. In other words, the second harmonic is not generated resonantly and this transverse propagation does not correspond to a waveguide mode; thus destructive interferences forbid the propagation of the second harmonic along the guide. This latter point is analogous to the case of an infinite medium.

4.4. The zero harmonic or mean-value field

Using (32), the equations (27) reduce to

$$\partial_z H_2^{0,x} = 0 \tag{44}$$

$$\partial_z \left(H_2^{0,z} + M_2^{0,z}\right) = 0 \tag{45}$$

$$m \left(H_2^{0,z} - \alpha M_2^{0,z}\right) = 0 \tag{46}$$

$$\alpha M_2^{0,y} - \Xi_2^{0,z} = 0. \tag{47}$$

Together with the continuity conditions at $z = L$ and $z = 0$, with $\vec{H}_2^0$ vanishing in the regions a and b, this yields

$$\vec{H}_2^0 = 0 \qquad M_2^{0,y} = \frac{-2\omega k q}{\alpha(\alpha^2 m^2 - \omega^2)} |g|^2 f_2 \qquad M_2^{0,z} = 0. \tag{48}$$

The $M_2^{0,x}$ component is not determined at all at this order. The equation accounting for this component is the x-component of the Landau–Lifschitz equation (3) at the following order ε^3. Making use of expressions (22), (23) of H_1^1, M_1^1, it reads as

$$V \partial_\xi M_2^{0,x} = \Xi_3^{0,x} = \frac{-\mathrm{i}\omega m q g \varphi_1}{\alpha^2 m^2 - \omega^2} H_2^{1,z,*} + q g \varphi_1 M_2^{1,y,*} + \text{c.c.} \tag{49}$$

Therefore, the computation of this nonlinear term involves the expressions of $H_2^{1,z}$, $M_2^{1,y}$, which are only obtained at the following order (formulae (61), (62)). These expressions are reported in (49). The terms involving ψ cancel each other, as do the terms involving $\mathrm{e}^{\mathrm{i}qz}$ and the terms involving $z\varphi_2$. Then $\Xi_3^{0,x}$ can be reduced to an exact derivative. Assuming that both g and $M_2^{0,x}$ vanish as $\xi \longrightarrow +\infty$, equation (49) can be integrated and

$$M_2^{0,x} = \frac{-(\alpha^2 m^2 + \omega^2) X}{\omega \alpha m V} f_1 |g|^2 . \tag{50}$$

The ratio X/V is given by formulae (119) below. With regard to the bulk situation (taking retardation into account), the computation and the expression of this zero-harmonic term are relatively simple.

5. The group velocity

The expression for the group velocity is again found when computing the term at the fundamental frequency, at the second order of the perturbative scheme.

5.1. The 'secular' terms are no longer forbidden

The equations for the fundamental frequency term, in the region a outside the magnetic film, at order ε^2, yield the following inhomogeneous system:

$$\partial_z H_2^{1,x} - \mathrm{i}k H_2^{1,z} = \mathrm{i}\partial_\xi A_1^1 \mathrm{e}^{-kz} \tag{51}$$

$$\mathrm{i}k H_2^{1,x} + \partial_z H_2^{1,z} = -\partial_\xi A_1^1 \mathrm{e}^{-kz}. \tag{52}$$

The general solution of these equations is written, for $z > L$, as

$$\vec{H}_2^1 = \left(A_2^1 + \mathrm{i}A_{1,\xi}^1 z\right) \begin{pmatrix} 1 \\ 0 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{-kz}. \tag{53}$$

$A_{1,\xi}^1$ is the ξ-derivative of the function $A_1^1 = A_1^1(\xi, \tau)$ introduced at order 1. The latter can be computed; it reads $A_1^1 = \mathrm{i}\eta k^2 g \mathrm{e}^{kL}$ ($\eta = \pm 1$), but in fact this expression is not needed to achieve the computation. A_2^1 is a function of (ξ, τ) to be determined. Because the rhs member of system (51), (52) is a solution of the homogeneous equation, the solution contains a term proportional to z. In the usual multiscale expansions, such solutions grow linearly and are called secular. They are not allowed because the field must remain bounded. Here this is not the case due to the decreasing exponential that ensures the vanishing of the 'secular' solution at infinity. Thus, properly speaking, the word 'secular' does not apply to this case, but we still use it for convenience. This is an essential feature because a large part of the usual multiscale expansion lies on the removal of secularities. We will see that an analogous feature occurs inside the magnetic medium. In the case of propagation in a bounded domain, here waveguide, 'secular' solutions are allowed. In the region b the result is analogous.

In the region i, inside the magnetic film, the equations governing the evolution of the second-order component of the fundamental frequency wave field are given by the system (27) with the rhs member given by expressions (28). It is solved in the same way as the general system (13) in appendix A. This computation is detailed in appendix D.2. The component $H_2^{1,z}$ is a solution of the differential equation (14) with the rhs member Σ_2^1 given by

$$\Sigma_2^1 = \frac{-q^2\sigma_2}{k^2} g_\xi \, \varphi_1. \tag{54}$$

The constant σ_2 depends on the velocity V; it is given by equation (118) in appendix D.2.

Here, as outside the magnetic medium, the rhs member is the solution of the homogeneous equation. We search a particular solution of the form of a 'secular' solution $H_{2,s}^{1,z} = wz\varphi_2$. It is found that

$$H_{2,s}^{1,z} = \frac{q\sigma_2}{2k^2} g_\xi \, z\varphi_2. \tag{55}$$

Note again an essential characteristic of the present problem with regard to the same kind of expansion, but in an unbounded medium. When considering the latter situation, such a secular term, proportional to z, increases linearly up to infinity. Since the field must be bounded, the secular terms are removed yielding the condition giving the group velocity. Here, the medium is bounded; thus $H_{2,s}^{1,z}$ is bounded, even with the term proportional to z. Thus the condition giving the group velocity is not given by the non-secularity condition, as it used to be in multiscale expansions for the waves propagating in a infinite media, but will appear elsewhere. The other field components are expressed as the general solution in appendix A.

Their inhomogeneous parts read

$$H_{2,s}^{1,x} = \frac{X}{k} g_\xi \varphi_2 + \frac{\mathrm{i}\sigma_2}{2k} g_\xi z\varphi_1 \tag{56}$$

$$M_{2,s}^{1,y} = \frac{-\mathrm{i}\omega m q \sigma_2}{2(\alpha^2 m^2 - \omega^2)k^2} g_\xi z\varphi_2 - \frac{(\alpha^2 m^2 + \omega^2)X}{\omega\alpha m q} g_\xi \varphi_1 \tag{57}$$

$$M_{2,s}^{1,z} = \frac{\alpha m^2 q \sigma_2}{2(\alpha^2 m^2 - \omega^2)k^2} g_\xi z\varphi_2 - \frac{2\mathrm{i}X}{q} g_\xi \varphi_1. \tag{58}$$

5.2. The waveguide conditions

The boundary conditions, analogous to those written at order 1, are solved using the formulae derived in appendix B in a general frame. Details are given in appendix D.2. The compatibility condition for this system yields the following expression for the wave packet velocity:

$$V = \frac{-\alpha m^2 q^2 k L}{(k^2+q^2)\,\omega\,[L(k^2+q^2)+2k]}. \tag{59}$$

Taking the k-derivative of the two equations (19) and (21), which constitute the dispersion relation, it is proved that V is the group velocity, $V = \frac{d\omega}{dk}$, of the wave. Expression (59) is well known. It gives the group velocity of a magnetostatic backward volume spin wave.

Then the complete expression of the field components at this order is found. It reads

$$H_2^{1,x} = -\mathrm{i}k\psi\, \varphi_2 + \frac{X}{k} g_\xi \varphi_2 + \frac{\mathrm{i}\sigma_2}{2k} g_\xi z\varphi_1 + \frac{\mathrm{i}qX}{k+\mathrm{i}q} g_\xi \mathrm{e}^{-\mathrm{i}qz} \tag{60}$$

$$H_2^{1,z} = q\psi\, \varphi_1 + \frac{q\sigma_2}{2k^2} g_\xi z\varphi_2 - \frac{\mathrm{i}q^2 X}{k(k+\mathrm{i}q)} g_\xi \mathrm{e}^{-\mathrm{i}qz} \tag{61}$$

$$M_2^{1,y} = \frac{-\mathrm{i}\omega m}{\alpha^2 m^2 - \omega^2}\left(q\psi\,\varphi_1 + \frac{q\sigma_2}{2k^2} g_\xi z\varphi_2 - \frac{\mathrm{i}q^2X}{k(k+\mathrm{i}q)} g_\xi \mathrm{e}^{-\mathrm{i}qz}\right) - \frac{(\alpha^2m^2+\omega^2)X}{\omega\alpha m q} g_\xi \varphi_1 \tag{62}$$

$$M_2^{1,z} = \frac{\alpha m^2}{\alpha^2 m^2 - \omega^2}\left(q\psi\,\varphi_1 + \frac{q\sigma_2}{2k^2} g_\xi z\varphi_2 - \frac{\mathrm{i}q^2X}{k(k+\mathrm{i}q)} g_\xi \mathrm{e}^{-\mathrm{i}qz}\right) - \frac{2\mathrm{i}X}{q} g_\xi \varphi_1. \tag{63}$$

Explicit expressions of the quantities A_1^1, A_2^1, etc, can be computed and give the expressions for the fields outside the medium. Since they are not needed in the following, we shall omit them.

6. The nonlinear evolution equation

6.1. The nonlinear term at third order

Outside the magnetic medium, the equations at order ε^3 at fundamental frequency are analogous to the previous order, equations (51) and (52), but with a slightly more complicated rhs member. Details are given in appendix E.1. In the upper region a, the field is given by

$$\vec{H}_3^1 = \left(A_3^1 + \mathrm{i}A_{2,\xi}^1\, z - \tfrac{1}{2}A_{1,\xi\xi}^1\, z^2\right)\mathrm{e}^{-kz}\begin{pmatrix}1\\0\\\mathrm{i}\end{pmatrix}. \tag{64}$$

Inside the nonlinear medium, the components of the fundamental frequency field at this order satisfy some system of the same type as (13). It involves a nonlinear term that divides into two parts: $\vec{\Xi}_3^1 = \vec{\Xi}_3^{1,(0)} + \vec{\Xi}_3^{1,(2)}$. The first term $\vec{\Xi}_3^{1,(0)} = \vec{M}_1^1 \wedge \vec{H}_2^0 + \vec{M}_2^0 \wedge \vec{H}_1^1$ gives an account of the self-interaction through the rectified field; the second one, $\vec{\Xi}_3^{1,(2)} = \vec{M}_1^{-1} \wedge \vec{H}_2^2 + \vec{M}_2^2 \wedge \vec{H}_1^{-1}$, gives an account of the self-interaction through the second harmonic that is called cascading in nonlinear optics. Not all the components of $\vec{\Xi}_3^1$ are needed, but only $\Xi_3^{1,x}$ and the combination $\mathrm{i}\omega\Xi_3^{1,z} - \alpha m\Xi_3^{1,y}$. The latter reads

$$\mathrm{i}\omega\Xi_3^{1,z} - \alpha m\Xi_3^{1,y} = \{\theta_1\,\varphi_2\,\sin\chi + \theta_2\,\varphi_1\,\cos\chi + \theta_3 f_2\varphi_2 + (\theta_4 f_1 + \theta_5)\varphi_1\}g|g|^2. \tag{65}$$

The variable χ is a shortcut for

$$\chi = q_2\left(z - \frac{L}{2}\right). \tag{66}$$

The explicit expressions of the coefficients θ_j are given in appendix E.2. The x-component reads

$$\Xi_3^{1,x} = [\rho_1\,\sin\chi + \rho_2 f_2]\varphi_1\,g|g|^2. \tag{67}$$

The term involving the coefficient ρ_1 in formula (67) gives an account of cascading, while the term involving ρ_2 gives an account partly of cascading and partly of the interaction with the zero harmonic or mean-value field.

6.2. Solution of a differential equation

The boundary conditions yield some linear inhomogeneous system of the same type as (20), whose compatibility condition is the nonlinear evolution equation sought for the wave amplitude g. An essential point of the derivation is to solve equation (14), satisfied by $H_1^{1,z}$, with the following expression for the rhs member:

$$\frac{-k^2}{q^2}\Sigma_3^1 = \sigma_1 g_\tau\,\varphi_1 + \sigma_2\psi_\xi\,\varphi_1 + \left[\sigma_3\,\varphi_1 + \sigma_4\,z\varphi_2 + \sigma_5\,\mathrm{e}^{-\mathrm{i}qz}\right]g_{\xi\xi} + \left[\sigma_6\,\varphi_2\,\sin\chi + \sigma_7\,\varphi_1\,\cos\chi\right.$$
$$\left. + \sigma_8\,\varphi_1^3 + \sigma_9\,\varphi_1\varphi_2^2 + \sigma_{10}\,\varphi_1\right]g|g|^2. \tag{68}$$

The expressions for the coefficients σ_j, together with details on the computation, are given in appendix E.3. The inhomogeneous part $H_{3,s}^{1,z}$ of $H_3^{1,z}$ is then computed. It reads as

$$H_{3,s}^{1,z} = \frac{q\sigma_1}{2k^2}\,z\varphi_2\,g_\tau + \frac{q\sigma_2}{2k^2}\,z\varphi_2\,\psi_\xi + \left[\frac{q\sigma_3}{2k^2}\,z\varphi_2 - \frac{\sigma_4}{4k^2}\left(q\,z^2\varphi_1 + z\varphi_2\right) + \frac{q\sigma_5}{2\mathrm{i}k^2}\,z\mathrm{e}^{-\mathrm{i}qz}\right]g_{\xi\xi}$$
$$+ \left[Q_1\,\varphi_2\,\sin\chi + Q_2\,\varphi_1\,\cos\chi + \frac{q(k^2+q^2)}{8k^2}(3\sigma_8 + \sigma_9)z\varphi_2\right.$$
$$\left. + \frac{\sigma_8 - \sigma_9}{8k^2}\left(f_1 - \frac{k^2+q^2}{2}\right)\varphi_1 + \frac{q\sigma_{10}}{2k^2}\,z\varphi_2\right]g\,|g|^2. \tag{69}$$

Then, with some computational work detailed in appendix E.4, the compatibility condition for the boundary conditions can be written explicitly. The term containing ψ_ξ vanishes. After division by an adequate quantity, the compatibility condition reduces to the following evolution equation for the field amplitude g:

$$\mathrm{i}g_\tau + Bg_{\xi\xi} + Cg|g|^2 = 0. \tag{70}$$

As expected, equation (70) is the NLS equation.

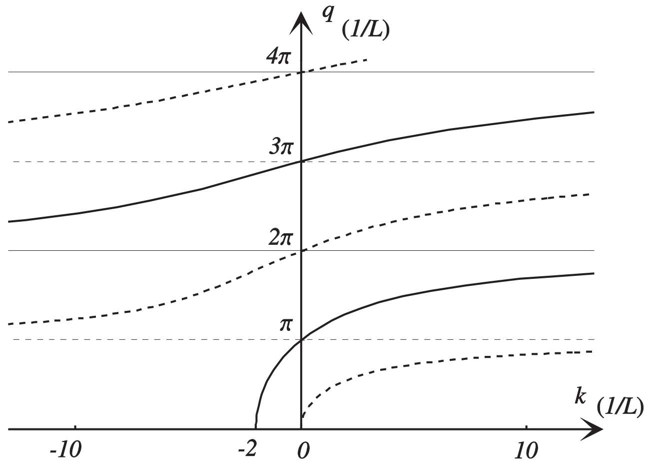

Figure 2. Plot of the waveguide dispersion relation. The unit is the inverse $1/L$ of the film thickness.

7. The coefficients of the NLS equation

7.1. The dispersion coefficient is not the group velocity dispersion

After reduction, the expression of the dispersion coefficient B of the NLS equation (70) reads as

$$B = \frac{m}{k^2}\frac{kL(\tilde{\omega}^2 - \alpha(1+\alpha))}{2\tilde{\omega}^3(\alpha(2(1+\alpha)+kL) - 2\tilde{\omega}^2)^3}[16\tilde{\omega}^2(\alpha(1+2\alpha) - 2\tilde{\omega}^2)(\tilde{\omega}^2 - \alpha(1+\alpha))^2$$
$$+\alpha k^2L^2(\alpha^2 - \tilde{\omega}^2)(\alpha^3(1+\alpha) + 2\alpha(1+\alpha)\tilde{\omega}^2 - 3\tilde{\omega}^4)$$
$$+2kL(\alpha^2 - \tilde{\omega}^2)(\alpha(1+\alpha) - \tilde{\omega}^2)(\alpha^3(1+\alpha) + 2\alpha\tilde{\omega}^2 - \tilde{\omega}^4)] \quad (71)$$

where $\tilde{\omega} = \omega/m$. The so-called 'method of envelopes', used in [10] to derive the NLS equation, states that this coefficient is $\omega''/2 = \frac{1}{2}\frac{d^2\omega}{dk^2}$, where the function $\omega(k)$ is the dispersion relation. This statement seems to be fully correct from the mathematical point of view in bulk media, but here in a very thin film we have a different result. We compare numerically the coefficient B given by equation (71) to $\omega''/2$, where $\omega(k)$ is defined by equations (19) and (21).

The waveguide dispersion relation (21) can be solved. Considering k as a function of q, there are two solutions

$$k = -q\cot\frac{qL}{2} \quad (72)$$

$$k = q\tan\frac{qL}{2}. \quad (73)$$

Relations (72) and (73) are transcendent and cannot be inverted explicitly. They are plotted in figure 2. The relation (72) is the solid curve and (73) is the dashed curve. They will be referred to as branch I and branch II, respectively, below. Apart from the material and field characteristics m and α, the parameters of wave are ω, k, q and the waveguide thickness L. These four parameters are related to each other by relations (19) and (21). Making use of the parameters k and qL allows the explicit solution of the dispersion relation. Figures 3 (branch I) and 4 (branch II) give the plots of B and $\omega''/2$ against qL for a fixed value of k. Due to homogeneity the curve does not depend on this value if the unit for B and $\omega''/2$ is m/k^2.

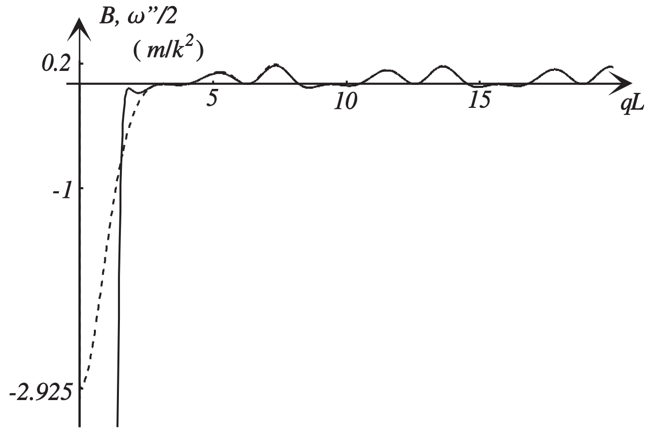

Figure 3. Plot of B (solid curve) and $\omega''/2$ (dashed curve) against qL for a constant value of k, for branch I of the dispersion relation. The field strength parameter is $\alpha = 1$.

It is seen that B is very close to $\omega''/2$, except if qL is small, say $qL \leqslant 5$. And when qL is very small, the two quantities do not have the same order of magnitude any more. On branch I in figure 3, $\omega''/2$ vanishes much faster than B. On branch II in figure 4, B becomes infinite while $\omega''/2$ tends to a finite limit. Looking at q and L as functions of qL shows that the small values of qL correspond to the first waveguide modes. The modes can be labelled using the following limiting case: for large values of L, the transverse wave vector q tends to the limit $n\pi/L$, n being an arbitrary integer, odd when q satisfies (72) and even when q satisfies (73). The integer n can be used to label the modes even if L is not large. Then, the small values of qL correspond to $n = 0$ or 1. As qL tends to 0, L remains bounded: its limiting value is $-2/k$ on branch I and 0 on branch II. Thus, the situations where qL is small can arise only if the value of the film thickness L is small enough. A plot of B and $\omega''/2$ against L (figure 5) clarifies this statement: they almost coincide for any mode except the first one of each type. An important discrepancy arises close to the value $L = -2/k$ where B diverges while $\omega''/2$ remains finite, which has already been observed in figure 4. An approximation of the involved quantities can be computed as k approaches $-2/L$. The dispersion relations are solved as

$$\omega \simeq \alpha m \qquad q \simeq \sqrt{\frac{6(k+2/L)}{L}}. \tag{74}$$

The dispersion coefficients read

$$B \simeq \frac{-m}{2L(k+2/L)^3} \qquad \omega''/2 \simeq \frac{-9(5+8\alpha)L^2 m}{160\alpha}. \tag{75}$$

Formula (75) confirms the observation made in figures 3 and 5(d). On the other hand, B and $\omega''/2$ coincide as qL tends to infinity,

$$B \sim \omega''/2 \sim \frac{m\sqrt{\alpha}\,\sin^2 qL[16\alpha - 2 + 8(1+4\alpha)\eta\cos qL - 6\cos 2qL]}{32\sqrt{2}\,k^2(1+2\alpha-\eta\cos qL)^{3/2}} \tag{76}$$

where $\eta = +1$ on branch I and $\eta = -1$ on branch II. The coefficient B that accounts for the dispersion in the NLS equation (70) is thus very close to the commonly admitted expression $\omega''/2$, except when the product qL takes very small values. This situation corresponds to the first two waveguide modes and appears mainly in very thin films (L small).

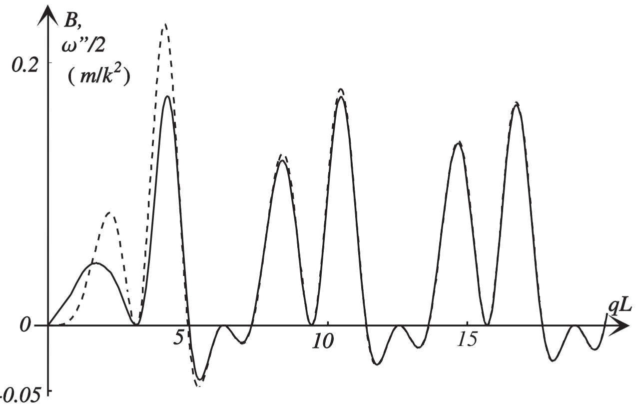

Figure 4. Plot of B (solid curve) and $\omega''/2$ (dashed curve) against qL for a constant value of k, for branch II of the dispersion relation. The field strength parameter is $\alpha = 1$.

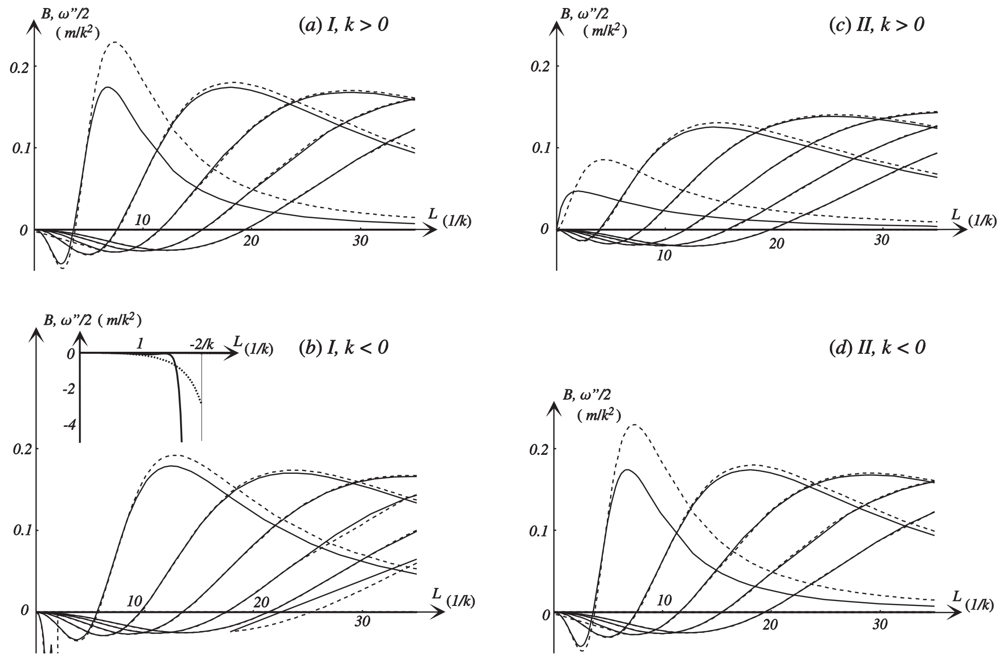

Figure 5. Plot of B (solid curve) and $\omega''/2$ (dashed curve) against L for constant value of k. (a) and (b) present branch I of the dispersion relation; (c) and (d) present branch II. The field strength parameter is $\alpha = 1$.

7.2. The nonlinear coefficient

The nonlinear coefficient C in equation (70) can be explicitly computed using the formulae listed in this paper, but its expression cannot be reduced to a reasonable length. A value has been given for this coefficient by Slavin and Rojdestvenski [11] using the description of the MSW spectrum of [12]. In order to compare both results, we need to use the same normalization of the wave amplitude. In [11], the function Ψ that obeys the NLS equation is

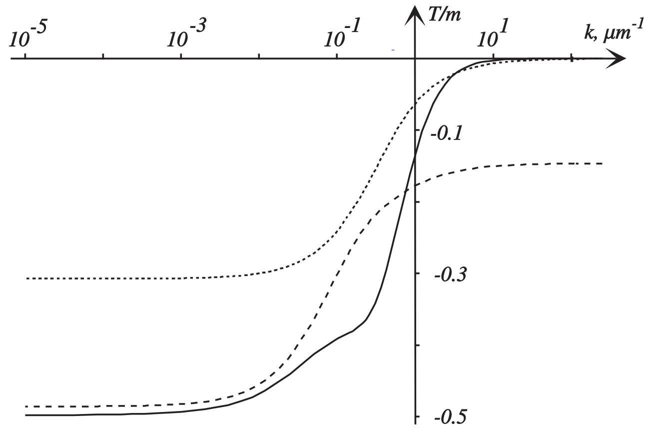

Figure 6. Comparison of the values of the normalized nonlinear coefficient T/m (or N/m) of the NLS equation according to the 'method of envelopes' (dotted curve), the work by Slavin and Rodjestvenski (dashed curve) and the present paper (solid curve). Parameter values are $\alpha = 1066/1750$ and $L = 7.2\ \mu$m.

'a dimensionless magnetization'. Using the notation of the present paper, and according to both equation (3) in [11] and the scaling (5), it satisfies

$$|\Psi|^2 = \frac{\left\|\vec{M}_1^1\right\|^2}{2m^2}. \tag{77}$$

According to the expression (107) of φ_1 and assuming that their average squared values are equal, we can identify the function $\Phi_n(\xi)$ in formula (21) of [11] as $\frac{\sqrt{2}\varphi_1}{\sqrt{k^2+q^2+2k/L}}$ and have $|\Psi| = |\varphi_n \Phi_n|$. The nonlinear factor $C|g|^2$ in equation (70) can be identified as the nonlinear factor $-T|\varphi_n|^2$ in equation (49) of [11]. We obtain

$$T = -C\frac{4(\alpha^2m^2 - \omega^2)^2}{q^2(k^2+q^2+2k/L)(\omega^2+\alpha^2m^2)}. \tag{78}$$

The value of the coefficient T obtained in this paper is numerically compared to that of [11], for the waveguide mode 0, as shown in figure 6. The explicit expression of the latter reads as

$$T_0 = \frac{m\,\mathrm{e}^{kL}}{32\alpha k^2L^2(\mathrm{e}^{kL} - 1 + \alpha kL\,\mathrm{e}^{kL})}\Bigg(\mathrm{e}^{-3kL}(\mathrm{e}^{kL}-1)\left(\sqrt{2}(\mathrm{e}^{kL}-1) + \left(2-\sqrt{2}\right)kL\,\mathrm{e}^{kL}\right) \times (\mathrm{e}^{kL} - 1 + 2\alpha kL\,\mathrm{e}^{kL}) - 4\mathrm{e}^{-4kL}(\mathrm{e}^{kL}-1)^2(\mathrm{e}^{2kL} - 1 - 2kL\,\mathrm{e}^{2kL}) + 4kL(\mathrm{e}^{-kL} - 1 - 2\alpha kL)^2\left(\frac{\mathrm{e}^{-kL} - 1 + kL}{\sqrt{2}\,kL} - 1\right)\Bigg). \tag{79}$$

A less rigorous approximation, the 'method of envelopes', gives another value of this coefficient, denoted by N in [11]. For the mode 0, it reads as

$$N = \frac{-\alpha m(1-\mathrm{e}^{-kL})}{kL\sqrt{\left(2\alpha\frac{1-\mathrm{e}^{-kL}}{kL}\right)^2 - \left(\frac{1-\mathrm{e}^{-kL}}{kL}\right)^2}}. \tag{80}$$

The parameter values in figure 6 are the same as in figure 1 of [11]. Our results are in good agreement with those of [11] for very small values of k. Surprisingly, they become much closer

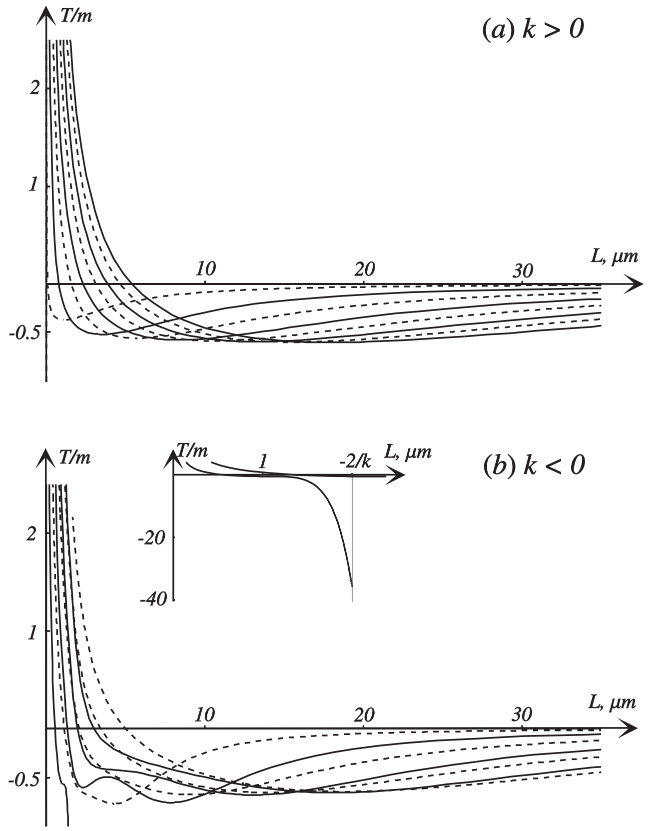

Figure 7. Plot of the normalized nonlinear coefficient T/m against the film thickness L. Solid curve; branch I, dashed curve; branch II. (a) Wave vector k is positive; $k = 1$ μm. (b) Wave vector k is negative; $k = -1$ μm. External field: $\alpha = 1$.

to that of the 'method of envelopes' for larger values of k. Note that we have neglected the inhomogeneous exchange interaction and the corresponding exchange boundary conditions taken into account in [11, 12]. We have set the corresponding parameters denoted by α and d, respectively, to 0 in the quoted references for comparison. In the YIG films, in the range where the discrepancy between both the theories is important, inhomogeneous exchange cannot be neglected.

The dependence of the normalized nonlinear coefficient T with regard to the film thickness L can be determined in the same way as the dispersion coefficient. We make use of the expressions (72) and (73) that solve explicitly the waveguide dispersion relation (21) to draw T against L using a parametric representation with the parameter qL. The result is given in figure 7. It is seen that T takes large positive values when the film thickness tends to 0 and negative values in the range of $-0.5 \times m$ to 0 for values of L of a few micrometres. A much larger negative value of T is obtained at the limit $L = -2/k$.

8. Conclusion

The nonlinear evolution equation for one-dimensional 'temporal' solitons in magnetic thin films is derived in a rigorous way using a multiscale expansion. The study was restricted to propagation parallel to the constant applied field (MS BVW) neglecting the inhomogeneous exchange and anisotropy. The asymptotic model is naturally the NLS equation, but the coefficients differ from the values computed by less rigorous approaches. A very remarkable feature is that the dispersion coefficient differs from the k-derivative of the group velocity. This occurs for the two lowest order modes when the transverse wave vector is small compared to the inverse of the film thickness and conversely. The nonlinear coefficient also differs from the values computed using other theories, but the difference is rather small for some typical common experimental values for which inhomogeneous exchange is negligible.

Until now, nonlinear evolution equations for envelope solitons (i.e. NLS-type equations) have been derived by means of the rigorous method of multiscale expansions in bulk media only. The waveguide properties are taken into account for almost the first time in the present paper. I say almost, because the same problem has been solved simultaneously for an optical waveguide filled with a Kerr material by my co-workers [13]. The same particularities appear in both cases, especially regarding the dispersion, which is in fact a linear description problem of the wave packet. An appreciable discrepancy between the so-called 'group velocity dispersion' and the dispersion coefficient of the NLS equation appears for very thin magnetic films. Many technical features are analogous, especially those I called 'transversal secular terms' above. In magnetic films the nonlinear process is quadratic. Therefore, some particular features arise in the behaviour of the second harmonic: while in bulk media only a non-propagating second harmonic term can have a significant interaction with the fundamental at the considered scales, in thin films, another second harmonic term arises and interacts. It propagates in the sense that its wave vector is the proper one for a second harmonic. But the interaction is still non-resonant because this second harmonic does not belong to a waveguide mode.

Acknowledgment

The author thanks Professor A N Slavin for fruitful scientific discussion.

Appendix A. Solution of the general system (13)

An adequate linear combination of the last two equations eliminates M^y and yields

$$M^z = \frac{\alpha m^2}{\alpha^2 m^2 - p^2\omega^2} H^z + \Lambda \tag{81}$$

with

$$\Lambda = \frac{\mathrm{i} p\omega U^z - \alpha m U^y}{\alpha^2 m^2 - p^2\omega^2}. \tag{82}$$

Taking the z-derivative of the first equation and subtracting from it $\mathrm{i}pk$ times the second equation eliminates H^x. Substituting in the obtained equation the above expression of M^z yields the differential equation (14) for H^z. The rhs member of this equation has the following expression:

$$\Sigma = \frac{q_p^2}{p^2k^2}\left(\partial_z^2\Lambda - \partial_z T^x + \mathrm{i}pkT^z\right). \tag{83}$$

Denoting by H_s^z some particular solution of equation (14) we obtain the expression (16) of H^z. Making use of the above expression (81) of M^z yields

$$M^z = \frac{\alpha m^2}{\alpha^2 m^2 - p^2\omega^2}(R\mathrm{e}^{\mathrm{i}q_p z} + S\mathrm{e}^{-\mathrm{i}q_p z}) + M_s^z \tag{84}$$

where

$$M_s^z = \frac{\alpha m^2}{\alpha^2 m^2 - p^2\omega^2} H_s^z + \Lambda. \tag{85}$$

H^x is deduced from the first equation of the system (13)

$$H^x = \frac{pk}{q_p}(R\mathrm{e}^{\mathrm{i}q_p z} - S\mathrm{e}^{-\mathrm{i}q_p z}) + H_s^x \tag{86}$$

$$H_s^x = \frac{T^x}{\mathrm{i}pk} - \frac{\mathrm{i}pk}{q_p^2}\partial_z H_s^z - \frac{1}{\mathrm{i}pk}\partial_z \Lambda. \tag{87}$$

M^y is computed from the last equation of the system (13)

$$M^y = \frac{-\mathrm{i}p\omega m}{\alpha^2 m^2 - p^2\omega^2}(R\mathrm{e}^{\mathrm{i}q_p z} + S\mathrm{e}^{-\mathrm{i}q_p z}) + M_s^y \tag{88}$$

$$M_s^y = \frac{-\mathrm{i}p\omega m}{\alpha^2 m^2 - p^2\omega^2} H_s^z + \frac{\alpha m U^z + \mathrm{i}p\omega U^y}{\alpha^2 m^2 - p^2\omega^2}. \tag{89}$$

Appendix B. Resolution of a second general system

B.1. The system

We intend to solve the following system that appears when writing the boundary conditions at each order and for any harmonic,

$$\begin{aligned} &B = \frac{pk}{q_p}(R - S) + J_0 \qquad -\mathrm{i}B = \frac{-p^2k^2}{q_p^2}(R + S) + K_0 \\ &A\mathrm{e}^{-pkL} = \frac{pk}{q_p}(R\mathrm{e}^{\mathrm{i}q_p L} - S\mathrm{e}^{-\mathrm{i}q_p L}) + J_L \qquad \mathrm{i}A\mathrm{e}^{-pkL} = \frac{-p^2k^2}{q_p^2}(R\mathrm{e}^{\mathrm{i}q_p L} + S\mathrm{e}^{-\mathrm{i}q_p L}) + K_L. \end{aligned} \tag{90}$$

Elimination of A and B reduces system (90) to

$$\begin{aligned} &(pk - \mathrm{i}q_p)R + (pk + \mathrm{i}q_p)S = \frac{q_p^2}{pk}(K_0 + \mathrm{i}J_0) \\ &(pk + \mathrm{i}q_p)\mathrm{e}^{\mathrm{i}q_p L}R + (pk - \mathrm{i}q_p)\mathrm{e}^{-\mathrm{i}q_p L}S = \frac{q_p^2}{pk}(K_L - \mathrm{i}J_L). \end{aligned} \tag{91}$$

The determinant of this linear 2×2 system reads as

$$D_p = 2\mathrm{i}\left[\left(q_p^2 - p^2k^2\right)\sin q_p L - 2pkq_p \cos q_p L\right]. \tag{92}$$

B.2. Non-invertible case

At order ε^1 the system is homogeneous. The trivial solution $R_1^1 = S_1^1 = 0$ is excluded; therefore, we must have $D_1 = 0$. This yields the dispersion relation (21). The general solution of (91) for $p = 1$ can then be written as

$$R = (\mathrm{i}q + k)\frac{qg}{2\mathrm{i}} \qquad S = (\mathrm{i}q - k)\frac{qg}{2\mathrm{i}} + S_s \tag{93}$$

where g is an arbitrary complex number and

$$S_s = \frac{q^2}{k(k + \mathrm{i}q)}(K_0 + \mathrm{i}J_0). \tag{94}$$

The normalization is chosen in order to simplify the expressions below.

The general solution (16), (84), (86) and (88) of system (13) for $p = 1$ can then be written as

$$H_x = -\mathrm{i}kg\varphi_2 - \frac{k}{q}S_s\mathrm{e}^{-\mathrm{i}qz} + H_s^x \tag{95}$$

$$H_z = qg\varphi_1 + S_s\mathrm{e}^{-\mathrm{i}qz} + H_s^z \tag{96}$$

$$M^y = \frac{-\mathrm{i}\omega m}{\alpha^2m^2 - \omega^2}\left(qg\varphi_1 + S_s\mathrm{e}^{-\mathrm{i}qz}\right) + M_s^y \tag{97}$$

$$M^z = \frac{\alpha m^2}{\alpha^2m^2 - \omega^2}\left(qg\varphi_1 + S_s\mathrm{e}^{-\mathrm{i}qz}\right) + M_s^z \tag{98}$$

where φ_1 and φ_2 are defined by equations (24) and (25).

The compatibility condition of system (91) reads

$$\begin{vmatrix} k - \mathrm{i}q & \frac{q^2}{k}(K_0 + \mathrm{i}J_0) \\ (k + \mathrm{i}q)\mathrm{e}^{\mathrm{i}qL} & \frac{q^2}{k}(K_L - \mathrm{i}J_L) \end{vmatrix} = 0. \tag{99}$$

Using formula (105) below, it reduces to

$$(K_L - \mathrm{i}J_L) + \eta(K_0 + \mathrm{i}J_0) = 0. \tag{100}$$

Expressions can also be given for A and B, but they are not useful for pursuing the computation.

B.3. Other terms

For $p \neq \pm 1$, the determinant D_p is *a priori* not 0. We assume that it never vanishes, which means that none of the waveguide modes of the higher harmonics coincide with the chosen mode of the fundamental. This is satisfied in real physical situations. Then system (91) admits a unique solution. If the additional condition

$$J_L = J_0 \qquad K_L = -K_0 \tag{101}$$

is satisfied, this solution reduces to

$$R = G_p\mathrm{e}^{-\mathrm{i}q_p\frac{L}{2}}(K_0 + \mathrm{i}J_0) \tag{102}$$

$$S = -G_p\mathrm{e}^{+\mathrm{i}q_p\frac{L}{2}}(K_0 + \mathrm{i}J_0) \tag{103}$$

where

$$G_p = \frac{2q_p^2}{pkD_p}\left[pk\,\cos q_p\frac{L}{2} - q_p\,\sin q_p\frac{L}{2}\right]. \tag{104}$$

Appendix C. Properties of some functions

C.1. Functions φ_1 and φ_2

These functions are defined above by (24) and (25). They give the z-dependence of the considered waveguide mode. We give here some of their useful algebraic properties. We have $\partial_z\varphi_1 = q\varphi_2$, $\partial_z\varphi_2 = -q\varphi_1$, $\varphi_1(0) = q$ and $\varphi_2(0) = k$. Expanding the expression (24) of $\varphi_1(L)^2$ and using the dispersion relation (21) yields $\varphi_1(L) = \eta q$ with $\eta = \pm 1$. An analogous procedure shows that $\varphi_2(L) = -\eta k$. It is easily seen that

$$(k + \mathrm{i}q)\,\mathrm{e}^{\mathrm{i}qL} = \varphi_2(L) + \mathrm{i}\varphi_1(L) = -\eta(k - \mathrm{i}q). \tag{105}$$

For branch I of the dispersion relation (21), expression (72), the expression of φ_1 can be factorized as

$$\varphi_1 = \frac{k}{\cos qL/2} \sin q\left(z - \frac{L}{2}\right). \tag{106}$$

Using expression (73), that is, for branch II, we get

$$\varphi_1 = \frac{k}{\sin qL/2} \cos q\left(z - \frac{L}{2}\right). \tag{107}$$

Note that $\cos qL/2 = \sin qL/2 = \frac{\pm k}{\sqrt{k^2+q^2}}$.

C.2. Functions f_1 and f_2

They are defined by $f_1 = \varphi_1^2$ and $f_2 = \varphi_1\varphi_2$, and appear in the computation of the nonlinear terms at second order. These functions can be re-expressed as follows. The dispersion relation (21) can be rewritten as

$$\frac{qk}{\sin qL} = \frac{q^2 - k^2}{2\cos qL} = \mu \tag{108}$$

where μ is some constant. Then

$$f_1 = \frac{q^2 + k^2}{2} + \mu \cos q(2z - L) \tag{109}$$

$$f_2 = -\mu \sin q(2z - L). \tag{110}$$

It follows from these expressions that $\partial_z f_1 = 2qf_2$, and so on. We also have $f_2(0) = qk = -f_2(L)$.

Appendix D. Some additional detail on the derivation

We give in this appendix some technical detail about the resolution of the perturbative scheme that has been omitted in the text for the sake of clarity.

D.1. The second harmonic

D.1.1. Inhomogeneous term. The system (27) is solved using the formulae proved in appendix A for the general system (13). Λ_2^2 is given by equation (82), in which $U_2^{2,j} = \Xi_2^{2,j}$ for $j = y, z$, given by equation (31). It is written as

$$\Lambda_2^2 = \frac{\mathrm{i}mkq(2\omega^2 + \alpha^2 m^2)}{(\alpha^2 m^2 - 4\omega^2)(\alpha^2 m^2 - \omega^2)} f_2 g^2. \tag{111}$$

(Λ_2^2 denotes the value of Λ corresponding to the second order and second harmonic, and so on.) $H_2^{2,z}$ is a solution of equation (14) with the rhs member Σ_2^2 defined by equation (83) and given by $\Sigma_2^2 = \nu f_2$ where

$$\nu = \frac{-3\mathrm{i}q_2^2 q^3}{2k} \frac{\alpha^2 m^3 g^2}{(\alpha^2 m^2 - 4\omega^2)(\alpha^2 m^2 - \omega^2)}. \tag{112}$$

The expression (35) of the particular solution $H_{2,s}^{2,z}$ follows, provided that $q_2 \neq q$. The other field components are written in appendix A (equations (84)–(89)).

D.1.2. Homogeneous term. The equations giving an account of the boundary conditions at second order for the second harmonic are given by the system (90) in which $p = 2$, $A = A_2^2$, etc, and the rhs member is defined by

$$J_{2,z_0}^2 = H_{2,s}^{2,x}\Big|_{z=z_0} \qquad K_{2,z_0}^2 = H_{2,s}^{2,z} + M_{2,s}^{2,z}\Big|_{z=z_0} \qquad \text{for } z_0 = 0 \quad \text{and} \quad z_0 = L. \tag{113}$$

Using the above expressions (36)–(39) of $\vec{H}_2^2$, $\vec{M}_2^2$ and the properties of f_2 listed in appendix C, it is seen that $J_{2,0}^2 = J_{2,L}^2$ and $K_{2,L}^2 = -K_{2,L}^2$. Then R_2^2 and S_2^2 are computed using formulae (102) and (103) and read

$$R_2^2 = Y\mathrm{e}^{-\mathrm{i}q_2\frac{L}{2}}g^2 \tag{114}$$

$$S_2^2 = -Y\mathrm{e}^{\mathrm{i}q_2\frac{L}{2}}g^2 \tag{115}$$

where

$$Y = G_2\frac{\mathrm{i}mq^2}{4(\alpha^2m^2-\omega^2)\omega^2}[q^2\omega^2 - k^2(\alpha^2m^2+\omega^2)]. \tag{116}$$

G_2 is given by (104), and is reduced to yield the expression (42) for Y. This allows computation of the expressions (40) and (41) of the homogeneous parts of $H_2^{2,x}$ and $H_2^{2,z}$, respectively. $H_2^{2,z}$ has the expression (16) where the first term involving R_2^2 and S_2^2 is given by (41) and the particular solution $H_{2,s}^{2,z}$ of equation (14) is given by (36) yielding formula (43). This achieves the computation of the second harmonic at second order.

D.2. The fundamental frequency term at second order

Inside the magnetic film, the evolution of this component is governed by system (27), with a rhs member given by expressions (28). For $p = 1$, $\vec{\Xi}_2^1$ is zero and using (22), (23), it reduces to

$$\begin{pmatrix} T_2^{1,x} \\ T_2^{1,z} \\ U_2^{1,y} \\ U_2^{1,z} \end{pmatrix} = \begin{pmatrix} \mathrm{i}kg_\xi\varphi_2 \\ qg_\xi\varphi_1 \\ \mathrm{i}\omega\dfrac{mVq}{\alpha^2m^2-\omega^2}g_\xi\varphi_1 \\ -\alpha m\dfrac{mVq}{\alpha^2m^2-\omega^2}g_\xi\varphi_1 \end{pmatrix}. \tag{117}$$

System (27) is solved in the same way as the general system (13) in appendix A. Formula (55) gives a particular solution $H_{2,s}^{1,z}$ of equation (14) with the rhs member Σ_2^1 given by expression (54). The constant σ_2 involved in this formula reads

$$\sigma_2 = -2\mathrm{i}q(X+k) \tag{118}$$

where

$$X = \frac{\omega\alpha m^2q^2V}{(\alpha^2m^2-\omega^2)^2}. \tag{119}$$

The other field components are expressed like the general solution (84)–(89), with constants R_2^1 and S_2^1.

The boundary conditions yield the same kind of linear system as (20), but are inhomogeneous. It reads like (90) in appendix B with

$$A = A_2^1 + \mathrm{i}A_{1\xi}^1 L \qquad B = B_2^1. \tag{120}$$

The expression for the rhs member is completely analogous with equation (113). The compatibility condition (100) then reduces to

$$X = \frac{-kL(k^2+q^2)}{L(k^2+q^2)+2k}. \tag{121}$$

This yields expression (59) for the velocity V.

R_2^1 and S_2^1 are given by expression (93), where g is replaced by a new indeterminate function ψ, and where

$$S_{2,s}^1 = \frac{-\mathrm{i}q^2}{k(k+\mathrm{i}q)} X g_\xi \tag{122}$$

is computed using equations (94). Then the expressions (60)–(63) of the field components at this order are found from equations (95) to (98).

Appendix E. The third order of the perturbative scheme

E.1. The problem to be solved

Outside the magnetic medium, the rhs member of the equation analogous to (51) (in the upper region a) reads $\mathrm{i}\left(A_{2,\xi}^1 + \mathrm{i}A_{1,\xi\xi}^1 z\right)\mathrm{e}^{-kz}$. The solution is given by equation (64). There is an analogous expression for the lower region b.

Inside the nonlinear medium, the system satisfied by the components of the fundamental frequency field at this order is (13) with

$$\begin{aligned} T_3^{1,x} &= -\mathrm{i}kM_3^{1,x} - \partial_\xi\left(H_2^{1,x} + M_2^{1,x}\right) \qquad T_3^{1,z} = \partial_\xi H_2^{1,z} \\ U_3^{1,j} &= -V\partial_\xi M_2^{1,j} + \partial_\tau M_1^{1,j} + \Xi_3^{1,j} \qquad \text{for} \quad j = y, z. \end{aligned} \tag{123}$$

The compatibility condition (100) for the fundamental at this order gives the nonlinear evolution equation for the wave amplitude g. Because we are only interested in this evolution equation, let us look first at the boundary conditions. As at order 2, they yield a system of the same type as (90) with

$$A = A_3^1 + \mathrm{i}A_{2,\xi}^1 L - \tfrac{1}{2}A_{1,\xi\xi}^1 L^2 \qquad B = B_3^1 \tag{124}$$

and a rhs member defined by

$$J_{3,z_0}^1 = H_{3,s}^{1,x}\Big|_{z=z_0} \qquad K_{3,z_0}^1 = \frac{-k^2}{q^2}H_{3,s}^{1,z} + \Lambda_3^1\Big|_{z=z_0} \tag{125}$$

for $z_0 = 0$ and $z_0 = L$. The derivation is achieved after the explicit computation of $J_{3,0}^1$, $K_{3,0}^1$, $J_{3,L}^1$ and $K_{3,L}^1$.

E.2. Nonlinear term

We need to compute the following components of $\vec{\Xi}_3^1$: $\Xi_3^{1,x}$ and the combination $\mathrm{i}\omega\Xi_3^{1,z} - \alpha m\Xi_3^{1,y}$. The latter is computed using the expressions (36)–(39) and (40), (41) for the second harmonic term, and the expression (50) for $M_2^{0,x}$ and (48) for $M_2^{0,y}$. We obtain expression (65) in which the coefficients have the following values

$$\theta_1 = \frac{2mkY(\alpha^2m^2+2\omega^2)}{\alpha^2m^2-4\omega^2} \qquad \theta_2 = \frac{-4mkqY}{q_2} \tag{126}$$

$$\theta_3 = \frac{-\alpha m^2k^2q(\alpha^2m^2+3\omega^2)}{2\omega^2(\alpha^2m^2-\omega^2)} + \frac{2\omega^2k^2q}{\alpha(\alpha^2m^2-\omega^2)} \tag{127}$$

$$\theta_4 = \frac{-\alpha m^2 q}{2}\left(\frac{k^2}{\omega^2} + \frac{q^2}{\alpha^2 m^2 - \omega^2}\right) + \frac{-q(\alpha^2 m^2 + \omega^2)X}{\omega V} \tag{128}$$

$$\theta_5 = \frac{1}{4}(k^2 + q^2)m^2 q\left(\frac{\alpha k^2}{\omega^2} - \frac{q^2}{\alpha^2 m^2 - \omega^2}\right). \tag{129}$$

The x-component is computed in an analogous way and given by equation (67) in which the coefficients have the values

$$\rho_1 = \frac{2q\omega m(\alpha^2 m^2 + 2\omega^2)Y}{(\alpha^2 m^2 - 4\omega^2)(\alpha^2 m^2 - \omega^2)} \tag{130}$$

$$\rho_2 = \frac{-\alpha m^2 k q^2}{2\omega(\alpha^2 m^2 - \omega^2)} + \frac{-2\omega k q^2}{\alpha(\alpha^2 m^2 - \omega^2)}. \tag{131}$$

E.3. Solution of equation (14) at third order

The equations of the perturbative scheme at this order are given by system (13) with the rhs member given by (123). In order to solve it, we use the formulae of appendix A and first compute Λ_3^1 defined by equation (82). Using the relations (23), (62), (63) and (65), it yields

$$\Lambda_3^1 = \frac{\sigma_1}{q^2}g_\tau\varphi_1 - \frac{2\mathrm{i}X}{q}\left[\psi_\xi\varphi_1 - \frac{\mathrm{i}qX}{k(k+\mathrm{i}q)}g_{\xi\xi}\mathrm{e}^{-\mathrm{i}qz} + \frac{\sigma_2}{2k^2}g_{\xi\xi}\varphi_2 z\right] + \lambda_1 g_{\xi\xi}\varphi_1 + \frac{1}{\alpha^2 m^2 - \omega^2}$$
$$\times\left[\theta_1 \sin\chi\ \varphi_2 + \theta_2 \cos\chi\ \varphi_1 + \theta_3 f_2\varphi_2 + \theta_4 f_1\varphi_1 + \theta_5\varphi_1\right]g\,|g|^2 \tag{132}$$

with

$$\sigma_1 = \frac{2\mathrm{i}qX}{V} \qquad \lambda_1 = \frac{-(\alpha^2 m^2 + 3\omega^2)VX}{(\alpha^2 m^2 - \omega^2)\omega q}. \tag{133}$$

Σ_3^1 is defined by equation (83). The first two terms in its expression are given by (123) and can be reduced using the first equation of system (27) for $p = 1$. The involved field components are given by relations (22), (61) and (67). After some reduction we get equation (68) in which

$$\sigma_3 = -q - \frac{2\mathrm{i}X\sigma_2}{k^2} + \lambda_1 q^2 \qquad \sigma_4 = \frac{\sigma_2^2}{2k^2} \qquad \sigma_5 = \frac{-\mathrm{i}qX\sigma_2}{k(k+\mathrm{i}q)} \tag{134}$$

$$\sigma_6 = -\frac{kq\rho_1}{\omega} + \frac{\theta_1\left(q^2 + q_2^2\right) + 2\theta_2 q q_2}{\alpha^2 m^2 - \omega^2} \tag{135}$$

$$\sigma_7 = -\frac{kq_2\rho_1}{\omega} + \frac{2\theta_1 q q_2 + \theta_2\left(q^2 + q_2^2\right)}{\alpha^2 m^2 - \omega^2} \tag{136}$$

$$\sigma_8 = \frac{kq\rho_2}{\omega} - \frac{(2\theta_3 - 3\theta_4)\,q^2}{\alpha^2 m^2 - \omega^2} \tag{137}$$

$$\sigma_9 = \frac{-2kq\rho_2}{\omega} + \frac{(7\theta_3 - 6\theta_4)\,q^2}{\alpha^2 m^2 - \omega^2} \qquad \sigma_{10} = \frac{q^2\theta_5}{\alpha^2 m^2 - \omega^2}. \tag{138}$$

$H_{3,s}^{1,z}$ is a particular solution of equation (14) with the rhs member Σ_3^1. Using the expressions of the z-derivatives of the functions φ_j and f_j, the elementary particular solutions of equation (14) can easily be computed. As an example, for $\Sigma = \varphi_1$, a solution is

$$H_s = \frac{-1}{2q}\varphi_2 z. \tag{139}$$

The resolution presents a difficulty when $\Sigma = \frac{-q^2}{k^2}\left(\sigma_8\varphi_1^3 + \sigma_9\varphi_1\varphi_2^2\right)$, because $\varphi_2^3 + \varphi_2\varphi_1^2 = (k^2+q^2)\varphi_2$ is a solution of the homogeneous equation. A particular solution H_s can be obtained from the form

$$H_s = w_1\left(\varphi_2^3 + \varphi_2\varphi_1^2\right)z + w_2\left(\varphi_1^3 - \varphi_1\varphi_2^2\right). \tag{140}$$

After the computation of w_1 and w_2 and making use of the identity $\varphi_1^2 - \varphi_2^2 = 2\mu\,\cos q\,(2z - L)$, the expression (140) of H_s reduces to

$$H_s = \frac{q(k^2+q^2)}{8k^2}(3\sigma_8 + \sigma_9)\varphi_2 z + \frac{\sigma_8 - \sigma_9}{8k^2}\varphi_1\left(f_1 - \frac{k^2+q^2}{2}\right). \tag{141}$$

A combination of these elementary particular solutions yields the expression (69) of $H_{3,s}^{1,z}$. The additional coefficients read as

$$Q_1 = \frac{q^2}{k^2q_2}\frac{q_2\sigma_6 - 2q\sigma_7}{q_2^2 - 4q^2} \qquad Q_2 = \frac{q^2}{k^2q_2}\frac{q_2\sigma_7 - 2q\sigma_6}{q_2^2 - 4q^2}. \tag{142}$$

E.4. End of the derivation

According to (125) the other quantity needed for the computation of the quantities $J_{3,0}^1, \ldots,$ that appear in the compatibility condition (100) is $H_{3,s}^{1,x}$ given by formula (87). It involves the z-derivatives of $H_{3,s}^{1,z}$ and Λ_3^1 given by (69) and (132), respectively. It has the form

$$\begin{aligned} H_{3,s}^{1,x} = {} & (\mu_1\,\varphi_2 + \mu_2\,z\varphi_1)g_\tau + (\mu_3\,\varphi_2 + \mu_4\,z\varphi_1)\psi_\xi + \left[\mu_5\,\varphi_2 + \mu_6\,z\varphi_1 + \mu_7\,z^2\varphi_2\right. \\ & \left. + (\mu_8 + \mu_9 z)\mathrm{e}^{-\mathrm{i}qz}\right]g_{\xi\xi} + \left[\mu_{10}\,\varphi_1\,\sin\chi + \mu_{11}\,\varphi_2\,\cos\chi + \mu_{12}\,\varphi_2\right. \\ & \left. + \mu_{13}\,z\varphi_1 + \mu_{14}\,\varphi_2\varphi_1^2 + \mu_{15}\,\varphi_2^3\right]g|g|^2 \end{aligned} \tag{143}$$

where the coefficients μ_j have relatively simple expressions, involving the quantities computed above (θ_j, σ_j, Q_j, etc), that we shall omit.

The quantities $J_{3,0}^1$, etc are given by expression (125). They can be written in the form

$$J_{3,0}^1 = k\mu_1 g_\tau + k\mu_3\psi_\xi + j_{1,0}g_{\xi\xi} + j_{2,0}\,g\,|g|^2 \tag{144}$$

$$K_{3,0}^1 = \frac{\sigma_1}{q}g_\tau - 2\mathrm{i}X\psi_\xi + \kappa_{1,0}g_{\xi\xi} + \kappa_{2,0}\,g\,|g|^2 \tag{145}$$

$$J_{3,L}^1 = -\eta(k\mu_1 - q\mu_2 L)g_\tau - \eta(k\mu_3 - q\mu_4 L)\psi_\xi + j_{1,L}g_{\xi\xi} + j_{2,L}\,g|g|^2 \tag{146}$$

$$K_{3,L}^1 = \frac{\eta\sigma_1}{2q}(2 + kL)g_\tau - \eta\left(2\mathrm{i}X - \frac{\sigma_2 kL}{2q}\right)\psi_\xi + \kappa_{1,L}g_{\xi\xi} + \kappa_{2,L}\,g\,|g|^2 \tag{147}$$

where the coefficients $j_{1,0}$, $j_{2,0}$, etc are easily computed using the previous formulae. The expression of the velocity V, or rather of the parameter X, ensures that the term containing ψ_ξ vanishes from the compatibility condition. The coefficient of the term g_τ is $-\mathrm{i}L(k^2 + q^2)/V$. After division by this quantity, the compatibility condition reduces to the NLS equation (70). The coefficients of this equation read as

$$B = \frac{-V}{L(k^2 + q^2)}[\kappa_{1,0} + \eta\kappa_{1,L} + \mathrm{i}j_{1,0} - \mathrm{i}\eta j_{1,L}] \tag{148}$$

$$C = \frac{-V}{L(k^2 + q^2)}[\kappa_{2,0} + \eta\kappa_{2,L} + \mathrm{i}j_{2,0} - \mathrm{i}\eta j_{2,L}]. \tag{149}$$

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